# CONSTRUCTION OF COORDINATE FUNCTIONS FOR SOLVING BOUNDARY-VALUE PROBLEMS IN HEAT CONDUCTION BY DIRECT METHODS

A. Akaev and A. Yu. Potyagailo

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A method is proposed for constructing coordinate functions, with the boundary conditions as well as the properties of the operator in the boundary-value problem taken into account, which will appreciably improve the accuracy of the first few approximations.

Direct methods are widely used for the practical solution of boundary-value problems in heat conduction [1-3]. Let it be required to determine the temperature field described by the equation

$$L[t] = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2} = -\frac{w(x, y, z)}{\lambda} \tag{1}$$

inside some region $\Omega$ and by the relation

$$t = t_0(x, y, z), \tag{2}$$

at the boundary S.

In order to obtain an approximate solution to the boundary-value problem (1)-(2) by a direct method, it is necessary to stipulate a priori an expression approximating the sought solution.

It is well known [1-3] that the choice of the approximating function affects strongly both the convergence of successive approximations and the complexity of subsequent calculations, making the applicability of direct methods largely dependent on the appropriate choice of the approximating function. As a rule, in practice one selects functions linearly dependent on some number of undetermined parameters, and among the most often used forms convenient for a large class of problem is [1]:

$$t_n = \omega(x, y, z) \sum_{k,m,s=0}^{n} a_{k,m,s} x^k y^m z^s + f(x, y, z) \quad (n = 0, 1, 2 \ldots), \tag{3}$$

where f(x, y, z) is an arbitrary function satisfying the nonhomogeneous condition (2) at the boundary S, $a_{k,m,s}$ are undetermined parameters, and $\omega(x, y, z)$ is a continuous function with bounded first partial derivatives inside region $\Omega$ and satisfying the conditions

$$\omega(x, y, z) > 0 \quad \text{inside} \quad \Omega, \; \omega(x, y, z) = 0 \quad \text{on} \quad S. \tag{4}$$

The corresponding system of coordinate functions

$$\omega(x, y, z) x^k y^m z^s \qquad (k, m, s = 1, 2 \ldots) \tag{5}$$

is relatively complete inside the given region $\Omega$ [1], which ensures that the successive approximations $t_n$ will converge at $n \to \infty$ to the exact solution to the original boundary-value problem (1)-(2), if parameters $a_{k,m,s}$ are determined according to the Ritz method or the Bubnov–Galerkin method. It must also be noted that (3) satisfies exactly the boundary condition (2) in the considered problem.

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The basic problem in stipulating the approximation in the form (3) is to construct the $\omega(x, y, z)$ function. The general rules for constructing it have been given in [1]. Recently Rvachev proposed an effective method of constructing the function $\omega(x, y, z)$ with the aid of the R-functions apparatus and logic algebra [4]. These known methods of constructing the $\omega(x, y, z)$ function do not account for the properties of the operator in the boundary-value problem, however, so that the successive approximations (4) converge slowly and a satisfactory accuracy requires high-order approximations. This has the following undesirable consequences: first of all, the solution becomes quite unwieldy and, secondly, the amount of calculations involved in finding the undetermined parameters increases fast. The authors here propose to use for function $\omega(x, y, z)$ the approximate solution to the problem according to the extended method by Kantorovich [5]. The function $\omega(x, y, z)$ constructed in this way will satisfy all stipulated requirements (4) and account for the properties of the operator in the boundary-value problem with respect to all variables, which contributes to a much faster convergence of the successive approximations (3) to the exact solution, while the accuracy needed in practical applications will be attained already in the first few approximations.

Example. Let it be required to integrate the equation

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = -1 \tag{6}$$

inside the rectangle $\Omega\,[-a, a; -b, b]$ with the condition that

$$t_0 = 0 \tag{7}$$

on its contour.

In [2] are given the first three approximations to the boundary-value problem (6)-(7) according to the Ritz method, with the function $\omega(x, y)$ selected in the conventional manner to have here the following form:

$$\omega(x, y) = (a^2 - x^2)(b^2 - y^2). \tag{8}$$

The said approximations are

$$t^{(m)} = (a^2 - x^2)(b^2 - y^2)\,[a_1^{(m)} + a_2^{(m)} x^2 + a_3^{(m)} y^2] \qquad (m = 1,\ 2,\ 3). \tag{9}$$

Thus, it is possible to estimate the effectiveness of the proposed method by selecting the function $\omega$, namely by comparing it with known solutions which use a function $\omega(x, y)$ of the form (8). It was on this basis that the boundary-value problem (6)-(7) had been selected for illustration.

We obtain an approximate solution to the problem (6)-(7) by the Ritz method, with the approximating expression in the form (3) and with function $\omega(x,y)$ constructed by the method just shown. For this purpose it is necessary, first of all, to solve the problem at hand by the extended Kantorovich method [5]. As result, we have

$$\tilde{t} = \frac{2\left(1 - \dfrac{\operatorname{th} p_x}{p_x}\right)\left(1 - \dfrac{\operatorname{ch} p_x \dfrac{x}{a}}{\operatorname{ch} p_x}\right)\left(1 - \dfrac{\operatorname{ch} p_y \dfrac{y}{b}}{\operatorname{ch} p_y}\right)}{p_y^2\left(2 - 3\dfrac{\operatorname{th} p_x}{p_x} + \dfrac{1}{\operatorname{ch}^2 p_x}\right)}. \tag{10}$$

Parameters $p_x$ and $p_y$ are related as follows:

$$\varepsilon p_x^2 = p_y^2 \frac{\operatorname{th} p_y}{p_y - \operatorname{th} p_y}, \qquad \varepsilon = \frac{a}{b}, \tag{11}$$

$$p_y^2 = \varepsilon p_x^2 \frac{\dfrac{\operatorname{th} p_x}{p_x} - \dfrac{1}{\operatorname{ch}^2 p_x}}{2 - 3\dfrac{\operatorname{th} p_x}{p_x} + \dfrac{1}{\operatorname{ch}^2 p_x}}. \tag{12}$$

Expression (10) exactly satisfies the original boundary condition (7) and all requirements (4) imposed on function $\omega$, so that expression (10) may be used for function $\omega$ without the constant factor. Thus,

$$\omega(x, y) = \left(1 - \frac{\operatorname{ch} p_x \dfrac{x}{a}}{\operatorname{ch} p_x}\right)\left(1 - \frac{\operatorname{ch} p_y \dfrac{y}{b}}{\operatorname{ch} p_y}\right). \tag{13}$$

TABLE 1. Numerical Calculations

| m | Values of t(0,0) and δt(0,0) | | | | |
|---|---|---|---|---|---|
| | exact | according to (17) | | according to (9) | |
| | $t(0,0)$ | $t^{(m)}(0,0)$ | $\delta t^{(m)}(0,0)$ % | $t^{(m)}(0,0)$ | $\delta t^{(m)}(0,0)$ % |
| 1 | 0,2930 | 0,2872 | —2,0 | 0,3125 | 6,9 |
| 2 | | 0,2924 | —0,2 | 0,3038 | 3,7 |
| 3 | | 0,2927 | —0,1 | 0,2922 | —0,3 |

The corresponding system of coordinate functions (5)

$$\{\varphi_{k,s}\} = \left\{\left(1 - \frac{\operatorname{ch} p_x \frac{x}{a}}{\operatorname{ch} p_x}\right)\left(1 - \frac{\operatorname{ch} p_y \frac{y}{b}}{\operatorname{ch} p_y}\right) x^k y^s\right\} \tag{14}$$

is relatively complete inside the stipulated region and, therefore, the sequence of approximations

$$t_n = \left(1 - \frac{\operatorname{ch} p_x \frac{x}{a}}{\operatorname{ch} p_x}\right)\left(1 - \frac{\operatorname{ch} p_y \frac{y}{b}}{\operatorname{ch} p_y}\right) \sum_{k,s=0}^{n} a_{k,s} x^k y^s \tag{15}$$

with coefficients $a_{k,s}$ found by the Ritz method or the Bubnov–Galerkin method will at $n \to \infty$ tend toward the exact solution to the boundary-value problem (6)-(7). Taking into account the symmetry of this problem, in the actual search for the approximate solution, we should express the approximating function (15) as

$$t_n = \left(1 - \frac{\operatorname{ch} p_x \frac{x}{a}}{\operatorname{ch} p_x}\right)\left(1 - \frac{\operatorname{ch} p_y \frac{y}{b}}{\operatorname{ch} p_y}\right) \sum_{p,q=0}^{n} a_{2p,2q} x^{2p} y^{2q}. \tag{16}$$

In order to realize how fast the successive approximations (16) converge and to estimate the accuracy of the first few approximations, we will consider the first three of them. These approximations are obtained by retaining the first three parameters in (16):

$$t^{(m)} = \left(1 - \frac{\operatorname{ch} p_x \frac{x}{a}}{\operatorname{ch} p_x}\right)\left(1 - \frac{\operatorname{ch} p_y \frac{y}{b}}{\operatorname{ch} p_y}\right)\left(a_{0,0}^{(m)} + a_{2,0}^{(m)} x^2 + a_{0,2}^{(m)} y^2\right) \tag{17}$$

$$(m = 1,\ 2,\ 3).$$

Here $a_{2,0}^{(1)} = a_{0,2}^{(1)} = a_{0,2}^{(2)} = 0$. Expressions for the remaining parameters are found from the Ritz equations. Numerical results have been obtained for the spherical case where $a = b = 1$ and the results are shown in Table 1. The values of $t^{(m)}(0,0)$ were calculated according to (9) and (17), using functions $\omega$ constructed by the conventional and by the proposed method, respectively. The relative errors in the values of the sought function

$$\delta t^{(m)}(0,\ 0) = \frac{t^{(m)}(0,\ 0) - t(0,\ 0)}{t(0,\ 0)}$$

were evaluated by comparing them with the exact value t(0,0) found from the solution in [1].

Solution (17) is more accurate than solution (9), according to Table 1, even though the example of a square region is the most inconvenient one for comparison, because in such a region the latter is most accurate while the former is least accurate.

With a function $\omega$ accounting for the properties of the operator in the boundary-value problem, therefore, the approximating expression (3) leads to a faster convergence of the successive approximations and to a better accuracy of the first few ones – which is a very important advantage from the practical standpoint.

## NOTATION

$t(x, y, z)$ is the sought temperature field;
$t^{(m)}$ is the m-th approximation of the sought temperature field;
$\omega(x, y, z)$ is the distribution density of energy sources;

x, y, z  are the Cartesian space coordinates;

λ  is the thermal conductivity.

## LITERATURE CITED


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